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applied to 0 in 0^{2k}. This is $\{2+\Delta^2(1+\Delta)^{-1}\}^\sigma 0^{2k}$; and its highest term has $\Delta^{2k}(1+\Delta)^{-k}0^{2k}$, in the development of which only $\Delta^{2k}0^{2k}$ has value. The term to be retained is therefore $k_\sigma 2^{\sigma-k}\Delta^{2k}0^{2k}$, or, σ being infinite,

$$\sigma^k 2^{\sigma-k}\Delta^{2k}0^{2k} \div 1.2\ldots k, \text{ or } \sigma^k 2^{\sigma-k} \times 1.3.5\ldots 2k-1 \times 2^k,$$

$\Delta^{2k}0^{2k}$ being $1.2.3\ldots 2k$. Divide by 2^σ, the number of drawings of $x+y+\ldots$ and we have $\sigma^k \times 1.3\ldots 2k-1$, in which σ, or $(1+1+1+\ldots\ \sigma$ terms), is the sum of the average squares of the drawings of $x, y, \ldots$

The theorem which contains the signification of $\sigma(\sigma-1)\ldots(\sigma-a-b-c-\ldots+1)$ divided by $[a.[b.[c.\ldots$ seldom or never finds a place in chapters on combination. I recommend it to the attention of the elementary student, who may find various demonstrations of it.

On the Rate of Interest in Loans repayable by Instalments. By PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

(Continued from page 102.)

MY second example is the Austrian Loan of 1865. The conditions are as follows:—

734,694 Bonds, each of £19. 17*s.* 0*d.*, issued at £13. 14*s.* 4*d.*, that is, at a discount of £6. 2*s.* 8*d.* each.

£1	0	0	per Bond to be paid on application, say Dec. 1, 1865, and the remaining £12. 14*s.* 4*d.* in the following instalments:—
1	19	7	on Dec. 15, 1865,
3	11	7	,, Feb. 10, 1866,
3	11	7	,, April 10, ,, ,
3	11	7	,, June 10, ,, ,
13	14	4	Total.

Subscribers will be at liberty to pay their Scrip in full on any one of the above dates, under discount at 6 per cent per annum.

The bonds are to bear interest at the rate of 9*s.* 11*d.* each (=·4958333) per half year, (a trifle under $2\frac{1}{2}$ per cent on the *nominal* amount) payment of which to become due on the 1st June and the 1st Dec. of each year.

The bonds are to be redeemed in thirty-seven years, 9928 (to be selected by lot) half-yearly, at the same dates as the payments of interest. The first drawing to take place in May, 1868, and the bonds then drawn to be paid off on the 1st of June thereafter.

Required the cost per cent of the loan to the borrower, and the rates realized on the bonds paid off each half year.

Here it will be observed that the number of the bonds being 734,694, while 9928 × 74 = 734,672, the payment of 22 bonds is left unprovided for. I shall assume that these 22 bonds are to be included in the last payment, making the number paid off at the end of the 78th half year, 9950.

The principal points in which this problem differs from the former are that here, first, the loan is issued at a discount; secondly, the repayments of principal do not commence immediately; and thirdly, that these repayments, (with the exception of the last—see above,) are uniform.

To determine the cost to the borrower we have to find the rate at which the values, at any epoch, of the sums receivable and payable by him are equal to each other. The most convenient epoch of reference, in this case as in the last, is the commencement of the transaction—the date of issue, Dec. 1st, 1865.

The receipts are reduced to this epoch as follows :—

Dec. 1 . .	734,694·000		
15 . .	1,454,081·875	14 days	3,346·380
Feb. 10 . .	2,629,592·275	71 „	30,690·584
Apr. 10 . .	2,629,592·275	130 „	56,194·027
June 10 . .	2,629,592·275	191 „	82,561·993
	10,077,552·700		172,792·984
	172,792·984		
	9,904,759·716		

The reduction is effected by discounting at 6 per cent, (the stipulated rate) the several instalments for the time to elapse between the epoch of reference and their respective dates of payment. The discount amounts to 172,793, deducting which from (£13. 14*s.* 4*d.* × 734,694 =) 10,077,553, the difference, 9,904,760, is the value at the chosen epoch of the borrower's receipts.

Next, to find the value of his payments at the same epoch. It will be convenient to consider these in three portions. The first is a uniform annuity, for four half-yearly terms, of the interest on the bonds; the second is a variable annuity deferred four terms, and to last for seventy-four, whose payments are, for each term, the sum of the interest on the principal unpaid at the beginning of that term and the amount of the bonds repayable at the end of it; and the third is the amount of the 22 residual bonds repayable at the end of the seventy-eighth term. I will deal with these in order.

First, the uniform annuity. This consists of four half-yearly payments of ·495833 × 734,694 = 364,285·775; and if we denote the required half-yearly rate by i, its value will be,

$$364{,}285\cdot775\,\frac{1-v^4}{i}.$$

Secondly, the variable annuity, to be entered upon in two years, and to make 74 half-yearly payments. I shall here use the theorem (1), which, adapted to this case, is,

$$v^4\left\{\frac{b_1}{i}+\frac{\Delta b_1}{i^2}+\ldots-v^{74}\left(\frac{b_{75}}{i}+\frac{\Delta b_{75}}{i^2}+\ldots\right)\right\}.$$

We shall determine b_1, &c., as follows:—

Int. on	734,694 bonds	@ ·495833	364,285·775
Payable	9,928 „	@ 19·85	197,070·800
			561,356·575 = b_1
Int. on	724,766 „	@ ·495833	359,363·142
Payable	9,928 „	19·85	197,070·800
			556,433·942 = b_2

It is unnecessary to go further. We see that b_2-b_1, or $\Delta b_1=-4922\cdot633$, which is the half-yearly interest on the 9928 bonds paid off the preceding year; and each succeeding payment will evidently have the same difference. Hence $\Delta^2 b_1$, &c. $=0$.

Hence also,

$$\begin{aligned} b_{75}&=b_1+74\Delta b_1,\\ &=561{,}356\cdot575-4922\cdot633\times74,\\ &=561{,}356\cdot575-364{,}274\cdot867,\\ &=197{,}081\cdot708, \end{aligned}$$

and $\Delta b_{75}=-4922\cdot633$.

Hence the value of this annuity is

$$v^4\left\{\frac{561{,}356\cdot6}{i}-\frac{4922\cdot633}{i^2}-v^{74}\left(\frac{197{,}081\cdot7}{i}-\frac{4922\cdot633}{i^2}\right)\right\}.$$

Thirdly, the value of the 22 bonds due at the end of the 78th half year is,

$$22\times19\cdot85\times v^{78}=436\cdot70v^{78}.$$

The sum of the three results, namely,

$$\frac{364{,}285{\cdot}8}{i}+\frac{197{,}070{\cdot}8v^4}{i}-\frac{4922{\cdot}633v^4}{i^2}-\frac{197{,}081{\cdot}7v^{78}}{i}+\frac{4922{\cdot}633v^{78}}{i^2}+436{\cdot}7v^{78},$$

or,

$$\frac{364{,}285{\cdot}8}{i}+\frac{197{,}070{\cdot}8}{i(1+i)^4}-\frac{4922{\cdot}633}{i^2(1+i)^4}-\frac{197{,}081{\cdot}7}{i(1+i)^{78}}+\frac{4922{\cdot}633}{i^2(1+i)^{78}}+\frac{436{\cdot}7}{(1+i)^{78}},$$

is the value of the borrower's payments at the epoch of reference, in terms of i. And we have now to find the value of i which makes this expression equal to 9,904,760. This is done by trial, and the operation is as follows:—

i	·04	·045	·0437	·0436473	·0436484
log i	$\bar{2}$·6020600	$\bar{2}$·6532125	$\bar{2}$·6404814	$\bar{2}$·6399574	$\bar{2}$·6399683
„ $(1+i)$	0·0170333	0·0191163	0·0185757	0·0185538	0·0185542
„ $(1+i)^4$	0·0681334	0·0764652	0·0743027	0·0742150	0·0742168
„ $i(1+i)^4$	$\bar{2}$·6701934	$\bar{2}$·7296777	$\bar{2}$·7147841	$\bar{2}$·7141724	$\bar{2}$·7141851
„ $i^2(1+i)^4$	$\bar{3}$·2722534	$\bar{3}$·3828902	$\bar{3}$·3552655	$\bar{3}$·3541298	$\bar{3}$·3541534
„ $(1+i)^{78}$	1·3286005	1·4910706	1·4489030	1·4471933	1·4472285
„ $i(1+i)^{78}$	$\bar{1}$·9306605	0·1442831	0·0893844	0·0871507	0·0871968
„ $i^2(1+i)^{78}$	$\bar{2}$·5327205	$\bar{2}$·7974956	$\bar{2}$·7298658	$\bar{2}$·7271081	$\bar{2}$·7271651

364285·8	5·5614423	5·5614423	5·5614423	5·5614423	5·5614423
	$\bar{2}$·6020600	$\bar{2}$·6532125	$\bar{2}$·6404814	$\bar{2}$·6399574	$\bar{2}$·6399683
A	6·9593823	6·9082298	6·9209609	6·9214849	6·9214740
197070·8	5·2946223	5·2946223	5·2946223	5·2946223	5·2946223
	$\bar{2}$·6701934	$\bar{2}$·7296777	$\bar{2}$·7147841	$\bar{2}$·7141724	$\bar{2}$·7141851
B	6·6244289	6·5649446	6·5798382	6·5804499	6·5804372
4922·633	3·6921974	3·6921974	3·6921974	3·6921974	3·6921974
	$\bar{3}$·2722534	$\bar{3}$·3828902	$\bar{3}$·3552655	$\bar{3}$·3541298	$\bar{3}$·3541534
C	6·4198440	6·3093072	6·3369319	6·3380676	6·3380440
197081·7	5·2946463	5·2946463	5·2946463	5·2946463	5·2946463
	$\bar{1}$·9306605	0·1442831	0·0893844	0·0871507	0·0871968
D	5·3639858	5·1503632	5·2052619	5·2074956	5·2074495
4922·633	3·6921974	3·6921974	3·6921974	3·6921974	3·6921974
	$\bar{2}$·5327205	$\bar{2}$·7974956	$\bar{2}$·7298658	$\bar{2}$·7271081	$\bar{2}$·7271651
E	5·1594769	4·8947018	4·9623316	4·9650893	4·9650323
436·7	2·6401832	2·6401832	2·6401832	2·6401832	2·6401832
	1·3286005	1·4910706	1·4489030	1·4471933	1·4472285
F	1·3115827	1·1491126	1·1912802	1·1929899	1·1929547

A	9107100	8095242	8336062	8346126	8345916
B	4211400	3672353	3800477	3805833	3805723
C	2629900	2038484	2172360	2178040	2177930
D	231200	141372	160421	161248	161231
E	144400	78470	91692	92276	92264
F	20	14	16	16	16
	13462920	11846079	12228247	12244251	12243919
	2861100	2179856	2332781	2339288	2339161
Result	10601820	9666223	9895466	9004963	9904758
	9904760	9904760	9904760	9904760	9904760
Error	697060	238537	9294	203	2

I commence my trials with $i = {\cdot}04$, conjecturably near the truth. It turns out to be too small, giving an error of 697060. I then try ·045, which is too great, error, 238537. I then use *False Position,* which, as applied to problems such as the present gives the following rule:—Multiply the error of the last result by the last correction of the rate, and divide the product by the difference of the last two results. The quotient will be a new correction to be applied in either augmentation or diminution of the rate last used, according as that rate was found to be too small or too great.

We thus have here

$$\frac{238537 \times {\cdot}005}{10601820 - 9666223} = \frac{1192{\cdot}685}{935597} = {\cdot}0013.^{*}$$

And ·045 − ·0013 = ·0437 is a corrected rate. This is found on trial to be still too great, but the error is reduced to 9294. For a new correction we have

$$\frac{9{\cdot}294 \times {\cdot}0013}{9895 - 9666} = \frac{{\cdot}01208}{229} = {\cdot}0000527.$$

Hence ·0437 − ·0000527 = ·0436473 is a second corrected rate, trial of which shows that it is a trifle too small, the error being 203.

It would hardly be considered necessary to seek a closer approximation than this. I have nevertheless, as a matter of curiosity, carried the operation a step further. Thus:—

$$\frac{203 \times {\cdot}0000527}{9904963 - 9895466} = \frac{{\cdot}01070}{9497} = {\cdot}0000011.$$

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* As the quotients in this operation are taken to only two or three places, it is unnecessary to use more than three or four figures in the numbers producing them. Thus, instead of the above we might have written

$$\frac{238{\cdot}5 \times {\cdot}005}{10602 - 9666} = \frac{1{\cdot}193}{936} = {\cdot}0013.$$

This gives $\cdot0436473+\cdot0000011=\cdot0436484$ for a third corrected rate, and trial of this rate gives, as shown, an error of 2.* It thus appears that the method employed enables us, by three or four trials, to assign the required rate to any degree of approximation that may be desired.

It is the half-yearly rate that has just been determined. The yearly rate is easily deduced from it thus:—

$$(1\cdot0436484)^2-1=\cdot0892018, \text{ the yearly rate;}$$

that is, 8·92018 per cent.

The foregoing loan was brought out in the latter end of 1865, and it gave rise to some correspondence in the *Times*. In the paper of Nov. 30 "A Broker," (under date Nov. 28,) gives his calculation of the rate somewhat as follows:—

Bonds 734,694 at £13. 14*s*. 4*d*. = £10,077,552. Annual interest £723,571 = 7·23 per cent per annum. Bonds to be paid off at £19. 17*s*. 0*d*. each, (an increase of 45 per cent,) in thirty-seven years from 1st June, 1868, in equal half-yearly payments. The amount paid for interest and the bonus on the principal will thus be the same as though the entire loan were retained for 18½ years from that date and then paid off.

Loan completed June, '66; interest to Dec. 1, '86, 20½ years at 9*s*. 11*d*. each bond per half year will be		14,935,716
Nominal Principal paid, 734,694 bonds, at £19. 17*s*. 0*d*. each . .	14,583,675	
Amount really borrowed . .	10,077,552	
		4,506,123
Amount paid for Interest and Bonus . .		19,441,839,

or at the rate of 9·40 per cent on the real amount of the loan.

The above is presented, I suppose, as only an approximation. I cannot say that I follow the author's reasoning; but it must be admitted that his result is, all things considered, wonderfully near the truth.

Dissatisfied apparently with "A Broker's" result, "An Actuary" takes up the matter in the *Times* of Dec. 4. I have not the details of his calculation, but the following is his conclusion:—"I think it will be seen from the foregoing that the cost to the borrower is, in round numbers, 7½ per cent per annum." The only observation I make on this is, that it affords illustration of a remark made towards the commencement of this paper, that the approximate methods used in practice for the solution of such problems as the present give results entitled to but little confidence.

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* In making the trial it is in this case necessary to form $\log(1+i)$ to 9 places in order to have $\log(1+i)^{78}$ true in the seventh place.

It remains to determine the rates that will be realized on the several bonds, which rates are dependent upon the periods at which the bonds are respectively paid off. The value of the investment in respect of each bond on the 1st December, 1865, is (9,904,760÷734,694=) 13·48148, and if paid off at the end of the nth half year the investor receives a half-yearly payment of (9*s.* 11*d.*=) ·495833 during the term, and at the end of it 19·85. Calling, as before, i the required half-yearly rate, we have for the values of the repayments at the same epoch,

$$\frac{\cdot 495833(1-v^n)}{i}, \text{ and } 19\cdot 85v^n,$$

respectively. Hence the equation for determining i is

$$\frac{\cdot 495833(1-v^n)}{i}+19\cdot 85v^n=13\cdot 48148$$

or,
$$\frac{1-v^n}{i}+40\cdot 03362v^n=27\cdot 18954.$$

From this the value of i corresponding to each admissible value of n must, as in previous cases, be found by trial. To avoid unnecessary labour, and as a guide in our trials, find first the limits within which the values of i are contained. For $n=1$ the foregoing becomes

$$41\cdot 0336v=27\cdot 1895$$

or
$$27\cdot 1895(1+i)=41\cdot 0336;$$

from which we easily find $i=\cdot 50917$.

And if $n=\infty$ the expression becomes

$$\frac{1}{i}=27\cdot 1894,$$

whence
$$i=\cdot 03678.$$

These are wide limits. But observing that from the nature of the case the values must decrease rapidly from the greater limit as n increases, that is as the period of repayment of the principal becomes more remote, it seems probable that for $n=5$, (the first value with which we have to do,) 10 per cent, that is, $i=\cdot 1$, will be pretty near the truth. Trying this we get for result 28·6485. This value of i is too small. Trying ·11 we get 27·4533, and we are still under the mark. The error being 27·4533 − 27·1895 = ·2638, we find a correction thus :—

$$\frac{\cdot 2638\times\cdot 1}{28\cdot 6485-27\cdot 4533}=\cdot 00221,$$

which gives for a new value of i, ·11 + ·00221 = ·11221. Trying this we get for result 27·1983, which is quite near enough, the error being only = ·0088. I have computed a few cases, and consigned the results to the small table below; but it is unnecessary to give further details. The annual rate is found from the half-yearly rate by the formula, $(1+i)^2-1$.

n.	Half-yearly Rate per Cent.	Error.	Annual Rate per Cent.
5	11·221	+ ·0088	23·700
25	4·708	+ ·0008	9·635
45	4·062	− ·0022	8·288
65	3·849	− ·0019	7·845
78	3·783	− ·0055	7·709
Limit	3·678		7·490

On the Theory of Annuities Certain. *By* WILLIAM MATTHEW MAKEHAM, *Fellow of the Institute of Actuaries.*

THE interest with which I have perused Mr. Gray's very able paper in the last number of the *Journal*, has led me to give some consideration to the general theory of varying Annuities; and as the results of the investigation appear to me somewhat striking, I am induced to think that a brief paper on the subject may not be unacceptable to that section of the scientific world to which this *Journal* more particularly addresses itself.

Nearly all that has hitherto appeared in connection with this subject is comprised, I believe, in Chapters XV. and XVI. of Baily's work on Interest and Annuities.* Like everything which proceeded from the hand of that excellent writer, the matter is treated with skill and perspicuity; but owing to the prevailing want of appreciation of the importance of the inquiry (an importance which Mr. Gray has been the first to demonstrate), and the restricted scope of the object which the author had in consequence proposed to himself, the result is very far indeed from satisfactory. In proof of this assertion I need only mention the fact that

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* This remark is applicable only to Annuities Certain, as I find that Mr. Gray himself, in his work on Life Contingencies, has treated upon Life Annuities the successive payments of which are the several orders of figurate numbers. The connection between the values of Annuities of two successive orders is shewn by Mr. Gray to be expressed by the equation $i_x^{(m)}=i_x^{(m-1)}+vp_x i_{x+1}^{(m)}$, where $i_x^{(m)}$ denotes the value of an annuity of the mth order of figurate numbers, on a life aged x. In Mr. G.'s arrangement the payment due at the end of the first year is always unity.